

ON CONVEX POLYTOPES IN \mathbb{R}^d CONTAINING AND AVOIDING ZERO

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Abstract

The goal of this paper is to establish certain inequalities between the numbers of convex polytopes in \mathbb{R}^d “containing” and “avoiding” zero provided that their vertex sets are subsets of a given finite set S of points in \mathbb{R}^d . This paper is motivated by a conjecture about these quantities put forward by E. Boros and V. Gurvich in 2002.

Keywords: d -dimensional space, convex polytopes, zero-containing polytopes, zero-avoiding polytopes.

1 Introduction

The notions and facts used but not described here can be found in [1].

Let S be a finite set of points in \mathbb{R}^d , $X \subseteq S$, and $z \in \mathbb{R}^d \setminus S$.

A set X is called *z -containing set* (a *z -avoiding set*) if z is in the interior of the convex hull of X (respectively, z is not in the interior of the convex hull of X).

A z -containing set X is *minimal* if X has no proper z -containing subset.

A z -avoiding set X is *maximal in S* if S has no z -avoiding subset containing X properly.

Let $\mathcal{C}(S)$ and $\mathcal{A}(S)$ denote the sets of minimal z -containing and maximal z -avoiding subsets of S , respectively.

In 2002 E. Boros and V. Gurvich put forward the following interesting conjecture.

Conjecture 1.1 *Suppose that S is a finite set of points in \mathbb{R}^d , $z \in \mathbb{R}^d \setminus S$, and S is a z -containing set. Then $|\mathcal{A}(S)| \leq 2d |\mathcal{C}(S)|$.*

For an affine subspace F of \mathbb{R}^d , let $\dim(F)$ denote the affine dimension of F and $R(X)$ denote the minimal affine subspace in \mathbb{R}^d containing X .

A finite set S of points in \mathbb{R}^d is said to be *in a general position* if for every $X \subseteq S$, $|X| - 1 \leq d \Rightarrow \dim(R(X)) = |X| - 1$.

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We say that z is in a general position with respect to S if $\dim(R(X)) < d \Rightarrow z \notin R(X)$ for every $X \subseteq S$.

One of our main results is the following theorem.

Theorem 1.1 *Let S be a finite set of points in the d -dimensional space \mathbb{R}^d , $z \in \mathbb{R}^d \setminus S$. Suppose that z is in a general position with respect to S . Then $|\mathcal{A}(S)| \leq d |\mathcal{C}(S)| + 1$.*

This theorem was announced in [2] and its proof was presented at the RUTCOR seminar directed by E. Boros and V. Gurvich in June 2002.

Later L. Khatchian gave a construction providing for every $d \geq 4$ a counterexample (S, z) in \mathbb{R}^d to Conjecture 1.1 such that S is not in a general position. From Theorem 1.1 it follows that in all these counterexamples z is not in a general position with respect to S .

In this paper we also give some strengthenings of Theorem 1.1.

In [2] it is shown that if S is a finite set of points in the plane \mathbb{R}^2 , $z \in \mathbb{R}^2 \setminus S$, and S is a z -containing set, then $|\mathcal{A}(S)| \leq 3|\mathcal{C}(S)| + 1$, and so Conjecture 1.1 is true for the plane.

2 Some notions, notation, and auxiliary facts

Given a convex polytope P in \mathbb{R}^d , a face F of a polytope P is called a *facet* of P if $\dim(R(F)) = d - 1$. Obviously, $R(F)$ is a hyperplane; we call $R(F)$ a *facet hyperplane* of P . Let, as above, S be a finite set of points in \mathbb{R}^d , $X \subseteq S$, and $z \in \mathbb{R}^d \setminus S$. Let $s \in S$.

We will use the following notation:

- $\text{conv}(X)$ is the convex hull of X ,
- if P is a convex polytope, then $V(P)$ is the set of vertices of P and $v(P) = |V(P)|$,
- as above, $\mathcal{C}(S)$ is the set of minimal z -containing subsets of S ; also $\mathcal{C}_s(S)$ is the set of members of $\mathcal{C}(S)$ containing s ,
- as above, $\mathcal{A}(S)$ is the set of maximal z -avoiding subsets of S ,
- $\mathcal{A}^s(S)$ is the set of subsets A in S such that A is maximal z avoiding in S and $A \setminus s$ is maximal z -avoiding in $S \setminus s$, and so $A \in \mathcal{A}(S)$ and $A \setminus s \in \mathcal{A}(S \setminus s)$,
- $\mathcal{A}_s(S) = \mathcal{A}(S) \setminus \mathcal{A}^s(S) = \{X \in \mathcal{A}(S) : s \in X \text{ and } X \setminus s \notin \mathcal{A}(S \setminus s)\}$,
- $\mathcal{Simpl}(S)$ is the set of simplexes C such that z is an interior point of C and $V(C) \subseteq S$ and $\mathcal{Simpl}_s(S)$ is the set of simplexes C in $\mathcal{Simpl}(S)$ such that $s \in V(C)$.
- $\mathcal{Conv}(S) = \{\text{conv}(X) : X \in \mathcal{C}(S)\}$ and $\mathcal{Conv}_s(S)$ is the set of members of $\mathcal{Conv}(S)$ containing s ,
- $\mathcal{H}(S)$ is the set of hyperplanes H such that H is a facet hyperplane of a simplex in $\mathcal{Simpl}(S)$ and $\mathcal{H}_s(S)$ is the set of hyperplanes in $\mathcal{H}(S)$ containing s , and
- $\mathcal{F}(S)$ is the set of subsets T of S such that $|T| = d$ and $\text{conv}(T)$ is a face of a simplex in $\mathcal{Simpl}(S)$ and $\mathcal{F}_s(S)$ is the set of subsets of S in $\mathcal{F}(S)$ containing s .

Obviously, we have:

Lemma 2.1 *Let S be a finite set of points in \mathbb{R}^d . Then*

- (a1) $|\text{Conv}(S)| = |\mathcal{C}(S)|$,
- (a2) $|\mathcal{H}(S)| \leq |\mathcal{F}(S)| \leq (d+1)|\text{Smpl}(S)|$, and accordingly, $|\text{Conv}_s(S)| = |\mathcal{C}_s(S)|$,
- (a3) $|\mathcal{H}_s(S)| \leq |\mathcal{F}_s(S)| \leq d |\text{Smpl}_s(S)|$, and
- (a4) $|\mathcal{A}^s(S)| = |\mathcal{A}(S \setminus s)|$, and so $|\mathcal{A}(S)| - |\mathcal{A}(S \setminus s)| = |\mathcal{A}_s(S)|$.

We remind that z is in a general position with respect to S if $\dim(R(X)) < d \Rightarrow z \notin R(X)$ for every $X \subseteq S$.

We will use the following well known and intuitively obvious fact.

Lemma 2.2 *Let P be a convex polytope in \mathbb{R}^d and z a point in the interior of P . Suppose that z is in general position with respect to $V(P)$. Then there exists $X \subseteq V(P)$ such that $\text{conv}(X)$ is a simplex of dimension d and z is in the interior of $\text{conv}(X)$.*

Proof We prove our claim by induction on the dimension d . The claim is obviously true for $d = 1$. We assume that the claim is true for $d = n - 1$ and will prove that the claim is also true for $d = n$, where $n \geq 2$. Thus, P is a convex polytope in \mathbb{R}^n . Since z is in general position with respect to $V(P)$, clearly $z \notin V(P)$. Let $p \in V(P)$ and L the line containing p and z . Then there exists the point t in $L \cap P$ such that the closed interval pLt contains z as an interior point and t is not an interior point of P . In a plane language, pL is the ray going from point p through point z and t is the first point of the ray which is not the interior point of P . Then t belongs to a face F of P which is a convex polytope of dimension at most $n - 1$. Since z is in general position with respect to $V(P)$, the dimension of F is $n - 1$ and t is in a general position with respect to $V(F)$. By the induction hypothesis, there exists $T \subseteq V(F)$ such that $\text{conv}(T)$ is a simplex of dimension $n - 1$ and t is in the interior of $\text{conv}(T)$. Put $X = T \cup p$. Then $X \subseteq V(P)$, $\text{conv}(X)$ of dimension n is a simplex and z is in the interior of $\text{conv}(X)$. \square

Given $X, Y \subset \mathbb{R}^d$ and a hyperplane H , we say that H separates X and Y (or separates X from Y) if $X \setminus H$ and $Y \setminus H$ belong to different half-spaces of $\mathbb{R}^d \setminus H$.

It is easy to see the following:

Lemma 2.3 *Let A and A' be z -avoiding subsets of S . Then*

- (a1) *there exists a hyperplane separating A from z ,*
- (a2) *if there exists a hyperplane separating both A and A' from z , then $A \cup A'$ is also a z -avoiding subsets of S , and therefore*
- (a3) *if A and A' are maximal z -avoiding subsets of S and there exists a hyperplane separating both A and A' from z , then $A = A'$, and so*
- (a4) *a maximal z -avoiding subset A in S is uniquely defined by every hyperplane separating A from z .*

We also need the following simple facts.

Lemma 2.4 *Let z be in a general position with respect to S and $X \subseteq S$. Then the following are equivalent:*

- (a1) $z \in \text{conv}(X)$ and
- (a2) z is in the interior of $\text{conv}(X)$.

Proof Obviously, (a2) implies (a1). Suppose, on the contrary, (a1) does not imply (a2). Then z belongs to $\text{conv}(X')$ for some $X' \subseteq X$ with $\dim(R(X')) < d$. Therefore z is not in a general position with respect to S , a contradiction. \square

Lemma 2.5 *Let z be in a general position with respect to S . If C is a minimal z -containing subset of S , then $\text{conv}(C)$ is a simplex, and so $\text{Smpl}(S) = \text{Conv}(S)$.*

Proof (uses Lemmas 2.2 and 2.4). Since C is z -containing, there exists $X \subseteq C$ such that $z \in \text{conv}(X)$ and $\text{conv}(X)$ is a simplex. By Lemma 2.4, z is in the interior of $\text{conv}(X)$. Since C is minimal z -containing, clearly $C = X$. Therefore by Lemma 2.2, $\text{conv}(C)$ is a simplex. \square

Lemma 2.6 *Let z be in a general position with respect to S . Suppose that $X \subset S$ and $s \in S \setminus X$. Let L be the line in \mathbb{R}^d containing s and z . If $\dim(R(X)) \leq d - 2$, then $L \cap R(X) = \emptyset$.*

Proof Suppose, on the contrary, $L \cap R(X) \neq \emptyset$. Then $z \in R(X \cup s)$ and $\dim(R(X \cup s)) < d$. Then z is not in a general position with respect to S , a contradiction. \square

Lemma 2.7 *Let S be a z -containing set and z in a general position with respect to S . Let A be a maximal z -avoiding set in S and $s \in S \setminus A$. Then A has a subset T such that*

- (a1) $|T| = d$,
- (a2) T belongs to a facet of $\text{conv}(A)$,
- (a3) $\text{conv}(T \cup s)$ is a simplex of dimension d containing z as an interior point, and therefore
- (a4) $R(T)$ is a facet hyperplane of A separating A from z .

Proof (uses Lemmas 2.2 and 2.6). Let L be the line in \mathbb{R}^d containing s and z . Since $s \notin A$ and A is a maximal z -avoiding subset in S , clearly z is in the interior of $\text{conv}(A \cup s)$.

Let $I = L \cap \text{conv}(A \cup s)$. Since $\text{conv}(A \cup s)$ is a convex set, clearly I is a line segment sLr , where r is not an interior point of $\text{conv}(A \cup s)$. We claim that $r \in \text{conv}(A)$. Indeed, if not, then r belongs to a facet R of $\text{conv}(A \cup s)$ containing s . Since sLr is a convex set and $s, r \in R$, we have: $z \in sLr \subseteq R$, and so $z \in R$. It follows that z is not an interior point of $\text{conv}(A \cup s)$, a contradiction. Thus, $\text{conv}(A) \cap L \neq \emptyset$.

Since $s \notin A$, there exists a unique point t in L such that the closed interval sLt in L with the end-points s and t has the properties: z is in the interior of sLt and $\text{conv}(A) \cap sLt = t$. In a plane language, sL is the ray going from point s through point z and t is the first point of the ray belonging to $\text{conv}(A)$. By Lemma 2.6, t belongs to the interior of a facet F of $\text{conv}(A)$. Hence, by Lemma 2.2, F has a set T of d vertices such that $T \subset A$ and $\text{conv}(T)$ is a simplex of dimension $d - 1$ containing t as an interior point. Then $\text{conv}(T \cup s)$ is a simplex of dimension d containing z as an interior point, and so t is an interior point of $\text{conv}(T)$. \square

3 Main results

First we will prove a weaker version of our main result to demonstrate the key idea concerning the relation between the minimal z containing and maximal z -avoiding subsets of S .

Theorem 3.1 *Suppose that S is a z -containing set and z is in a general position with respect to S . Then $|\mathcal{A}(S)| \leq (d+1)|\mathcal{C}(S)|$.*

Proof (uses Lemmas 2.3(a4), 2.5, and 2.7). Let A be a maximal z -avoiding subset of S . Since S is a z -containing set, there exists $s \in S \setminus A$. Since A is a maximal z -avoiding subset of S , $A \cup s$ is not a z -avoiding subset of S . Hence $A \cup s$ is a z -containing set, and so z is in the interior of $A \cup s$. By Lemma 2.7, A has a subset T such that $|T| = d$, T belongs to a face of $\text{conv}(A)$, and $\text{conv}(T \cup s)$ is a simplex containing z as an interior point.

Let, as above, $\mathcal{H}(S)$ denote the set of hyperplanes H such that H is a facet hyperplane of a simplex in $\mathcal{Simpl}(S)$ and $\mathcal{F}(S)$ denote the set of subsets T of S such that $|T| = d$ and $\text{conv}(T)$ is a facet of a simplex in $\mathcal{Simpl}(S)$. Obviously, $|\mathcal{H}(S)| \leq |\mathcal{F}(S)| = (d+1)|\mathcal{Simpl}(S)|$. By Lemmas 2.3(a4) and 2.7, $|\mathcal{A}(S)| \leq |\mathcal{H}(S)|$. Clearly, $|\text{Conv}(S)| = |\mathcal{C}(S)|$ and, by Lemma 2.5, $\mathcal{Simpl}(S) = \text{Conv}(S)$. Thus, $|\mathcal{A}(S)| \leq (d+1)|\mathcal{C}(S)|$. \square

A hyperplane H in $\mathcal{H}(S)$ is said to be *essential* if H is a facet hyperplane of a maximal z -avoiding subset A in S separating A from z , and *non-essential*, otherwise. Let $\mathcal{H}^e(S)$ and $\mathcal{H}_s(S)$ denote the sets of essential hyperplanes in $\mathcal{H}(S)$ and $\mathcal{H}_s(S)$, respectively.

Lemma 3.1 *Let S be a finite set of points in \mathbb{R}^d , $z \in \mathbb{R}^d \setminus S$, and $s \in S$. Suppose that S is a z -containing set and z is in a general position with respect to S . Then*

$$|\mathcal{A}(S)| - |\mathcal{A}(S \setminus s)| = |\mathcal{A}_s(S)| \leq |\mathcal{H}_s^e(S)| \leq |\mathcal{H}_s(S)| \leq |\mathcal{F}_s(S)| \leq d |\mathcal{Simpl}_s(S)| = d |\mathcal{C}_s(S)|.$$

Proof (uses Lemmas 2.3(a4), 2.5, and 2.7). We prove that $|\mathcal{A}_s(S)| \leq |\mathcal{H}_s^e(S)|$. Let $A \in \mathcal{A}_s(S)$. Then $s \in A$ and $A' = A \setminus s$ is a z -avoiding but not maximal z -avoiding set in $S' = S - s$. Therefore there exists $s' \in S' \setminus A'$ such that $A' \cup s'$ is also a z -avoiding set. Obviously, $A' \cup \{s, s'\}$ is a z -containing set in S . By Lemma 2.7, A has a subset T such that $|T| = d$, T belongs to a face of $\text{conv}(A)$, and $\text{conv}(T \cup s')$ is a simplex containing z as an interior point. Since $A' \cup s'$ is a z -avoiding set, clearly $s \in T$. Now by Lemmas 2.3(a4) and 2.7, $|\mathcal{A}_s(S)| \leq |\mathcal{H}_s^e(S)|$. By Lemma 2.5, $\mathcal{Simpl}_s(S) = \text{Conv}_s(S)$, and clearly, $|\text{Conv}_s(S)| = |\mathcal{C}_s(S)|$. All the other inequalities in our claim are obvious. \square

Now we are ready to prove the following strengthening of Theorem 3.1 which is also an extension of Theorem 1.1.

Theorem 3.2 *Let S be a finite set of points in the d -dimensional space \mathbb{R}^d , $z \in \mathbb{R}^d \setminus S$. Suppose that z is in a general position with respect to S . Then*

- (a1) $|\mathcal{A}(S)| \leq d |\mathcal{C}(S)| + 1$,
- (a2) if either S is z -avoiding or S is z -containing and $|S| = d + 1$ (and so $\text{conv}(S)$ is a d -dimensional simplex), then $|\mathcal{A}(S)| = d |\mathcal{C}(S)| + 1$,
- (a3) if S is z -containing and $|S| \geq d + 2$, then $|\mathcal{A}(S)| \leq d |\mathcal{C}(S)| - d + 1$,
- (a4) if S is z -containing and $|S| = d + 2$, then $|\mathcal{A}(S)| = d |\mathcal{C}(S)| - d + 1$, and

(a5) if S is z -containing and $|S| \geq d + 3$, then $|\mathcal{A}(S)| \leq d |\mathcal{C}(S)| - d$.

Proof (uses Lemmas 2.5 and 3.1).

(p1) First we prove (a1). Our claim is obviously true if S is a z -avoiding set. Therefore we assume that S is a z -containing set. We prove our claim by induction on $|S|$. By Lemma 2.5, $|S| \geq d + 1$. If $|S| = d + 1$, then $\text{conv}(S)$ is a simplex, and our claim is obviously true. Thus, we assume that our claim is true for every z -containing set S with $|S| = k \geq d + 1$ and will prove that the claim is also true if $|S| = k + 1$. Since S is z -containing, by Lemma 2.5, there exists $X \subseteq S$ such that $\text{conv}(X)$ is a simplex and z is the interior point of $\text{conv}(X)$. Since $|X| = d + 1 < |S|$, there exists $s \in S \setminus X$. Obviously, $S' = S \setminus s$ is a z -containing set. Since $k = |S'| < |S| = k + 1$, by the induction hypothesis, our claim is true for $S' = S \setminus s$, i.e. $|\mathcal{A}(S')| \leq d |\mathcal{C}(S')| + 1$. By Lemma 3.1, $|\mathcal{A}(S)| - |\mathcal{A}(S')| = |\mathcal{A}_s(S)| \leq d |\mathcal{C}_s(S)|$. Now since $|\mathcal{C}(S)| = |\mathcal{C}(S')| + |\mathcal{C}_s(S)|$, our inductive step follows.

(p2) We prove (a2). If S is z -avoiding, then $|\mathcal{A}(S)| = 1$ and $|\mathcal{C}(S)| = 0$, and so $|\mathcal{A}(S)| = d |\mathcal{C}(S)| + 1$. If S is z -containing and $|S| = d + 1$, then $\text{conv}(S)$ is a d -dimensional simplex, z is in the interior of $\text{conv}(S)$, $|\mathcal{A}(S)| = d + 1$, and $|\mathcal{C}(S)| = 1$, and therefore $|\mathcal{A}(S)| = d |\mathcal{C}(S)| + 1$.

(p3) We prove (a3). Namely, we should prove that if S is a z -containing set and $|S| \geq d + 2$, then $|\mathcal{A}(S)| \leq d |\mathcal{C}(S)| - d + 1$. The proof of this inequality is similar to that in **(p1)**. We only need to prove the inequality for $|S| = d + 2$ to provide the beginning of induction. We will prove that if S is a z -containing set and $|S| = d + 2$, then $|\mathcal{A}(S)| = d |\mathcal{C}(S)| - d + 1$. Since S is z -containing, there exists $S' \subset S$ such that $\Delta = \text{conv}(S')$ is a simplex and z is in the interior of $\text{conv}(S')$, and so there is $s \in S$ such that $S' = S \setminus s$. Let L be the line containing s and z and sLt be the closed interval in L such that z is in the interior of sLt and t belongs to a face of Δ . Obviously, such interval exists. Let F be a face of Δ containing t . Since z is in a general position with respect to S , clearly t is an interior point of F and $\dim(R(F)) = d - 1$, and so $v(F) = d$. Let $s' = S' \setminus V(F)$. Then $\Delta' = \text{conv}(F \cup s) = \text{conv}(S \setminus s')$ is a z -containing simplex, and so $V(F) \cup s$ is a minimal z -containing subset of S . By the above definition, $\Delta = \text{conv}(S \setminus s) = \text{conv}(F \cup s')$ is another z -containing simplex, and so $V(F) \cup s' = S \setminus s$ is another minimal z -containing subset of S . On the other hand, from the definitions of sLt and F it follows that $S \setminus x$ is a z -avoiding subset of S for every $x \in V(F)$. Since S is a z -containing set, clearly each $S \setminus x$ is a maximal z -avoiding subset of S . Also $V(F)$ is a z -avoiding subset of S and since both $V(F) \cup s$ and $V(F) \cup s'$ are z -containing, clearly $V(F)$ is a maximal z -avoiding subset of S . Thus, $\mathcal{A}(S) = \{S \setminus x : x \in V(F)\} \cup \{V(F)\}$. Also both $\Delta = \text{conv}(F \cup s')$ and $\Delta' = \text{conv}(F \cup s)$ are z -containing simplexes, and so $\mathcal{C}(S) = \{S \setminus s, S \setminus s'\}$. Therefore, $d + 1 = |\mathcal{A}(S)| \leq d |\mathcal{C}(S)| - d + 1 = d + 1$. It follows that (a4) also holds.

(p4) The proof of (a5) is similar to that in **(p3)** because it can be checked that the inequality holds if $|S| = d + 3$. Claim (a5) is also a particular case of Theorem 3.3 below. \square

Our next goal is to prove a strengthening of Theorem 3.2 that takes into consideration the size of S .

Lemma 3.2 *Let S be a finite set of points in \mathbb{R}^d , $z \in \mathbb{R}^d \setminus S$, and z in a general position with respect to S . Let $P = \text{conv}(S)$ and $V = V(P)$. Suppose that $|S| \geq d + 2$ and S is a z -containing set. Then there exist a simplex Δ and $u \in V(\Delta)$ such that*

- (a1) $V(\Delta) \subseteq V$,
- (a2) z is an interior point of Δ ,
- (a3) Δ and P have a common facet hyperplane H , and
- (a4) $u \in H$ and $S \setminus u$ is a z -containing set.

Proof (uses Lemmas 2.2 and 2.4). We prove our claim by induction on d . It is easy to prove that our claim is true for $d = 2$. Therefore we assume that the claim is true for $d = n - 1$ and prove that it is also true for $d = n \geq 3$.

Since S is a z -containing set, z is an interior point of P . Let $v \in V$ and L be the line containing v and z . Let vLz' be a maximal segment in L such that $z \in vLz'$ and $vLz' \subset P$. Obviously, such segment exists (and is unique) and z' belongs to a face F of P . Moreover, since z is in a general position with respect to S , clearly F is a facet of P and point z' is in a general position with respect to $S' = V(F)$ (and so S' is a z' -containing set) in \mathbb{R}^{n-1} . By Lemma 2.2, F contains an $(n - 1)$ -dimensional simplex T such that z' is in the interior of T . Then $Q = \text{conv}(T \cup v)$ is an n -dimensional simplex satisfying (a1), (a2), and (a3) with $H = R(F) = R(T)$.

(p1) Suppose that $|S'| = |V(F)| > n$, and so $F \neq T$. Now by the induction hypothesis for (S', z') , there exists a simplex Δ' and $u \in V(\Delta')$, satisfying (a1) - (a4) in \mathbb{R}^{n-1} . Let $\Delta = \text{conv}(\Delta' \cup v)$. Then (Δ, u) satisfies (a1) - (a4) in \mathbb{R}^n .

(p2) Finally, suppose that $|S'| = |V(F)| = n$, and so $F = T$. We know that $|S| \geq n + 2$.

(p2.1) Suppose that P is a simplex. Then $|V(P)| = n + 1$. Let $s \in S \setminus V(P)$. Then s is in the interior of P . Obviously, there exists a facet F of P such that z is in the interior of $\text{conv}(F \cup s)$. Let u be a (unique) vertex of P not belonging to F and $\Delta = P$. Then (Δ, u) satisfies (a1) - (a4).

(p2.2) Finally, suppose that P is not a simplex. We remind that $F = T$. Then P has a facet F' such that $\dim(F \cap F') = n - 2$. Let $v' \in V(F') \setminus V(F)$, $Q' = \text{conv}(T \cup v')$, and $T' = \text{conv}((T \cap F') \cup v')$. Then Q' is a simplex and both T and T' are facets of Q' . Let $D = R(F)$ and $D' = R(F')$. Then $D = R(T)$ and $D' = R(T')$.

Suppose that $z \in Q'$. By Lemma 2.4, z is in the interior of Q' . Since z is in the interior of $Q \subset P \setminus v'$, clearly $S \setminus v'$ is z -containing. Let $\Delta = Q'$ and $u = v'$. Then (Δ, u) satisfies (a1) - (a4) with $H = D'$.

At last, suppose that $z \notin Q'$. Then Q has a facet C such that z is in the interior of the simplex $\text{conv}(C \cup v')$, and so $C \neq T$. Let $\Delta = Q$ and u be a (unique) vertex in $V(T) \setminus V(C)$. Then z is in the interior of $P \setminus u$, and therefore $S \setminus u$ is z -containing. Thus, (Δ, u) satisfies (a1) - (a4) with $H = D$. \square

Now we are ready to prove the following strengthening of Theorem 3.2.

Theorem 3.3 *Let S be a finite set of points in the d -dimensional space \mathbb{R}^d , $z \in \mathbb{R}^d \setminus S$. Suppose that z is in a general position with respect to S and $|S| \geq d + 2$. Then $|\mathcal{A}(S)| \leq d |\mathcal{C}(S)| - |S| + 3$.*

Proof (uses Lemmas 3.1 and 3.2 and Theorem 3.2). We prove our claim by induction on $|S|$. If $|S| = d + 2$, then by Theorem 3.2 (a3), the claim is true. We assume that our claim is true for $|S| = k \geq d + 2$ and prove that it is also true if $|S| = k + 1$. By Lemma 3.1,

$$|\mathcal{A}(S)| - |\mathcal{A}(S \setminus s)| = |\mathcal{A}_s(S)| \leq |\mathcal{H}_s^e(S)| \leq |\mathcal{H}_s(S)| \leq d |\mathcal{Simpl}_s(S)| = d |\mathcal{C}_s(S)|.$$

By Lemma 3.2, there exist a point u in S and a simplex Δ in $\mathcal{Simpl}_u(S)$ such that $S \setminus u$ is z -containing and $R(T) = R(F)$ for some facets T and F of Δ and $\text{conv}(S)$, respectively. Then obviously, $R(T)$ is a non-essential hyperplane in $\mathcal{H}_u(S)$, and therefore $|\mathcal{H}_u^e(S)| \leq |\mathcal{H}_u(S)| - 1$. Hence

$$|\mathcal{A}(S)| - |\mathcal{A}(S \setminus u)| = |\mathcal{A}_u(S)| \leq |\mathcal{H}_u^e(S)| < |\mathcal{H}_u(S)| \leq d |\mathcal{C}_u(S)|.$$

By the induction hypothesis, we have: $|\mathcal{A}(S \setminus u)| \leq d |\mathcal{C}(S \setminus u)| - |S \setminus u| + 3$.

Thus, our inductive step follows from the last two inequalities. \square

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